Nonlocal Quasipotential Equation in Terms of Retarded Functions

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A version of the quasipotential approach to the two-body problem in quantum electrodynamics is considered, formulated in terms of retarded functions of the Heisenberg field operators and representing a kind of the "new Tamm-Dankoff method". The quasipotential is shown to be renormalizable. An approximation is derived from the quasipotential equation which coincides with the Breit equation for two interacting particles.

I. Introduction

Several types of Schrödinger-like equations were proposed within the framework of the quasipotential approach [1] to the two-body problem in quantum field theory, which may be generally divided into two classes: local [2] and nonlocal [1, 3]. In the local versions one proceeds from an off-shell extrapolation of the two-particle scattering amplitude. The nonuniqueness of this extrapolation is then exploited by imposing additional requirements. The nonlocal versions, a particular kind of which this paper represents, are based on: *a*) equations of motion [4, 5, 6] or iterated Yang-Feldman formalism [7] (usually called "the new Tamm-Dankoff method"); *b*) equal-time Bethe-Salpeter equation [8]; c) reduction formulae for matrix elements of retarded [9, 10] or time-ordered [11, 12] products of the field operators. Unlike the local versions, the nonlocal equations are derived from the general structure of quantum field theory leaving no ambiguity for the quasipotential.

This work aims to present a consistent nonlocal formalism in spinor quantum electrodynamics, formulated in terms of retarded products and possessing the following properties: *a*) it is renormalizable within the framework of covariant renormalization theory for retarded functions [13]; *b*) it reproduces the well-known Breit formula for the energy levels of a two-body bound-state system with an accuracy of up to and including the order α^4. It is organized as follows: in Sec. 2 we apply the Nishijima reduction technique for retarded products of fields operators [9] to derive the bound-state quasipotential equation:

$$(1.1)\qquad \Phi_{OB}(\mathbf{x}_1, \mathbf{x}_2, t) = \int dt' d^3y_1 d^3y_2 \overline{L}(\mathbf{x}_1, \mathbf{x}_2, t; \mathbf{y}_1, \mathbf{y}_2, t')\, \Phi_{OB}(\mathbf{y}_1, \mathbf{y}_2, t'),$$

where $\Phi_{OB}(\mathbf{x}_1, \mathbf{x}_2, t)$ is the relativistic bound state wave function of two fermion fields $\psi_i(x)$ of spin 1/2, masses m_i, electric charges e_i, $i=1, 2$, interacting via electromagnetic field $A^\mu(x)$:

(1.2) $$\Phi_{OB}(\mathbf{x}_1, \mathbf{x}_2, t)=\langle 0 \,|\, \psi_1(x_1)\psi_2(x_2)\,|B\rangle|_{x_1^0=x_2^0=t},$$

$|B\rangle$ represents the bound state of the particles 1 and 2,[1] $\overline{L}$ is a sum over all equal-time irreducible time-ordered R-diagrams (Appendix A). The Green function of the equation turns out to be the four point generalized Nishijima retarded function [9] (see also Sec. 2), instead of the Green function of the equal-time Bethe-Salpeter [14] and Logunov-Tavkhelidze [1] equation which is the time-ordered vacuum expectation value $\langle 0 \,|\, T\psi_1(x_1)\psi_2(x_2)\overline{\psi}_1(y_1)\,\overline{\psi}_2(y_2)|0\rangle$. The quasi-potential equation is written in a Dirac-like form with a nonlocal quasipotential. The results so far agree with those of Baumann [10], who obtained the "new Tamm-Dankoff" quasipotential equation (1.1) in the case of one massive scalar field using the bound state reduction technique [9]. In Section 3 the quasipotential is shown to be renormalizable both in the case of two different-particle bound-state and for positronium. Section 4 is devoted to a local approximation of equation (1.1) which reproduces in the radiation-gauge quantum electrodynamics the Breit equation for hydrogen-like atoms and positronium [15] with an accuracy of up to the order α^4. The rules for time-ordered R-diagrams are formulated in Appendix A. In Appendix B the quasipotential (off-shell) scattering amplitude is shown to fit the relativistic Lippmann-Schwinger equation and to coincide with the physical two-particle scattering amplitude on the mass and energy shell.

2. Quasipotential Equation

We shall follow the notation and definitions of [13] and consider the wave function of the two-particle system (1.2). Applying the bound state reduction formulae [9] we obtain:

(2.1) $$\Phi_{OB}(\mathbf{x}_1, \mathbf{x}_2, x^0)=\int d^4y_1\, d^4y_2\, K_Y\langle 0\,|\, R(\psi_1(\mathbf{x}_1, x^0)\,\psi_2(\mathbf{x}_2, x^0);\; \overline{\psi}_1(y_1)\overline{\psi}_2(y_2))|0\rangle f_B(y_1, y_2)$$

(2.2a) $$f_B(y_1, y_2)=\exp\{iPY\}g_B(y)$$

(2.2b) $$Y=\mu_1 y_1+\mu_2 y_2,\; P=p_1+p_2;\quad \mu_j=\frac{(m_j^2+\mathbf{p}^2)^{1/2}}{w},\quad j=1, 2$$
$$y=y_1-y_2,\quad p=\mu_2 p_1-\mu_1 p_2;\quad m_j<w<m_1+m_2,$$

where (2.2a) denotes the free wave function of the bound state $|B\rangle$, (2.2b) are the "centre of mass" coordinates and momenta, $P^2=w^2$, w — mass of the bound state, $K_Y=\Box_Y+w^2$. We introduce the following notations:

$$\overset{\Gamma}{h}=h(x_1, x_2;\, y_1, y_2)|_{x_1^0=x_2^0};\; \overset{\neg}{h}=h(x_1, x_2;\, y_1, y_2)\,|_{y_1^0=y_2^0};\; \overline{h}=h(x_1, x_2;\, y_1 ,y_2)\,|_{x_1^0=x_2^0,\; y_1^0=y_2^0}.$$

The Green function of equation (2. 1) is the four-point Nishijima generalized retarded function [9] with equated times:

(2.3) $$\overline{G}(\mathbf{x}_1, \mathbf{x}_2 x^0;\; \mathbf{y}_1, \mathbf{y}_2, y^0)=\langle 0\,|\, \psi_1(x_1)R(\psi_2(x_2)\,\overline{\psi}_1(y_1)\overline{\psi}_2(y_2)) + R(\psi_1(x_1)\overline{\psi}_1(y_1)\overline{\psi}_2(y_2))\,\psi_2(x_2)-R(\psi_1(x_1)\,\overline{\psi}_1(y_1))R(\psi_2(x_2)\,\overline{\psi}_2(y_2))$$

[1] For a detailed discussion of the operation of "time-ordering" in (1.2) see, for example, [3].

$$-R(\psi_1(x_1)\psi_2(y_2))R(\psi_2(x_2)\bar{\psi}_1(y_1))|0\rangle|_{\substack{x_1^0=x_2^0=x^0\\ y_1^0=y_2^0=y^0}}.$$

To see this we consider the right hand side of (2.3) with different times:

$$G(x_1,x_2;y_1,y_2)=\langle 0|\psi_1(x_1)R(\psi_2(x_2)\bar{\psi}_1(y_1)\bar{\psi}_2(y_2))$$

$$(2.3')\quad +R(\psi_1(x_1)\bar{\psi}_1(y_1)\bar{\psi}_2(y_2))\psi_2(x_2)-R(\psi_1)(x_1)\bar{\psi}_1(y_1))R(\psi_2(x_2)\bar{\psi}_2(y_2))$$

$$-R(\psi_1(x_1)\psi_2(y_2))R(\psi_2(x_2)\bar{\psi}_1(y_1))|0\rangle.$$

If we insert a complete set of intermediate asymptotic states between the separate retarded products in (2.3) and then apply the reduction formulae of Lehmann-Symanzik-Zimmerman [16] we obtain as a result G written in terms of (ordinary) retarded functions. Hence, G is a well-defined tempered distribution in perturbation theory. It is constructed by the renormalization procedure for retarded products and retarded functions described in [13]. Graphically G is expressed (in perturbation theory) in terms of R-diagrams of the Dyson type [13] and satisfies a Bethe-Salpeter type of equation, which could be easily seen by graphical arguments:[1]

$$(2.4)\qquad G=G_0+\Lambda G=G_0+G_0'TG_0;\quad \Lambda=G_0'K$$

$$(2.4a)\quad G_0(x_1,x_2;y_1,y_2)=\overset{(1)}{S}_r(x_1-y_1)\overset{(2)}{S}_r(x_2-y_2);\ \overset{(j)}{S}_r(x)=\int\frac{d^4p}{(2\pi)^4}\frac{e^{-ipx}(\hat{p}+m_j)}{(m_j^2-p^2-ip^0 0)}$$

$$j=1,2,\quad G_0^{-1}GG_0^{-1}=G^{\mathrm{Amp}}=G_0^{-1}+T,$$

where $S_r(x)$ is the free retarded propagator, T is the "off-shell" scattering amplitude (see Appendix B), G_0' is the analytic expression corresponding to any of the products of two lines in Fig. 1, K is a sum over all two-fermion (many-time) irreducible R-diagrams of G with respect to the lines depicted in Fig. 1.

$$(2.4b)\qquad S^{(\pm)}(x)=\mp\frac{i}{(2\pi)^3}\int d^4pe^{-ipx}\theta(\mp p^0)\delta(p^2-m^2)(\hat{p}+m).$$

Now we are going to describe a graphical transition procedure from

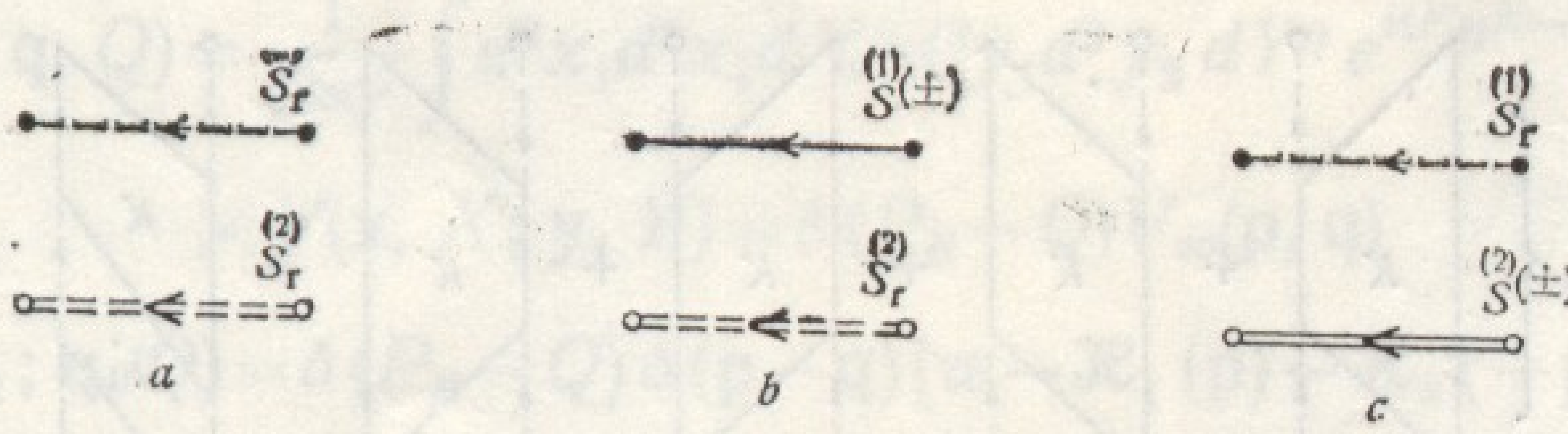


Fig. 1. All the graphs that correspond to G_0'

$\overset{\Gamma}{\Lambda}G$ towards $\bar{L}\overset{\Gamma}{G}$, where $\bar{L}$ is defined in Section 1 (see (1.1), based on successive iterations of (2.4) and "equal-time cuttings" (see Appendix A) of any of the two lines in G_0' (Fig. 2). In Fig. 2 time-ordering is imposed on the vertices

[1] Here and in the sequel we shall often suppress arguments and integral signs. Thus $G=G_0+\Lambda G$ should read explicitly:

$$G(x_1,x_2;y_1,y_2)=G_0(x_1,x_2;y_1,y_2)+\int d^4\xi_1 d^4\xi_2\Lambda(x_1,x_2;\xi_1,\xi_2)G(\xi_1,\xi_2;y_1,y_2).$$

$\xi_1, \eta_1, \xi_2, \eta_2; x_1, \xi_1, x_2, \xi_2; y_1, u_1, y_2, u_2 . S_1, S_2, S$ denote "equal-time cuttings". In diagrams I — IV in Fig. 2*a* we may perform "cutting" of the corresponding lines connecting Λ and G^{Amp}. Then the sum of those subgraphs that are situat-

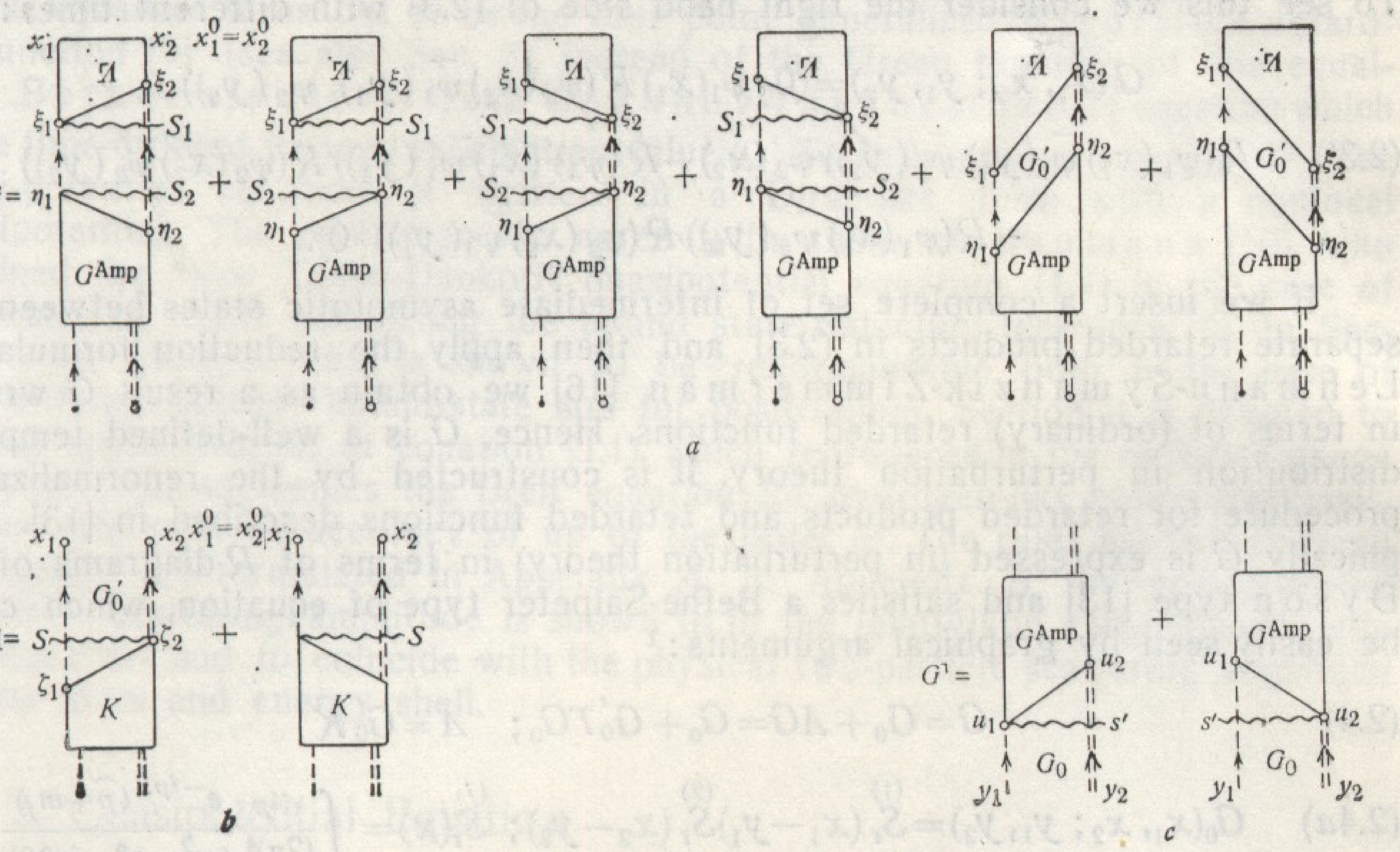


Fig. 2. Graphical transition procedure from ΛG towards $\overline{L}\overline{G}$

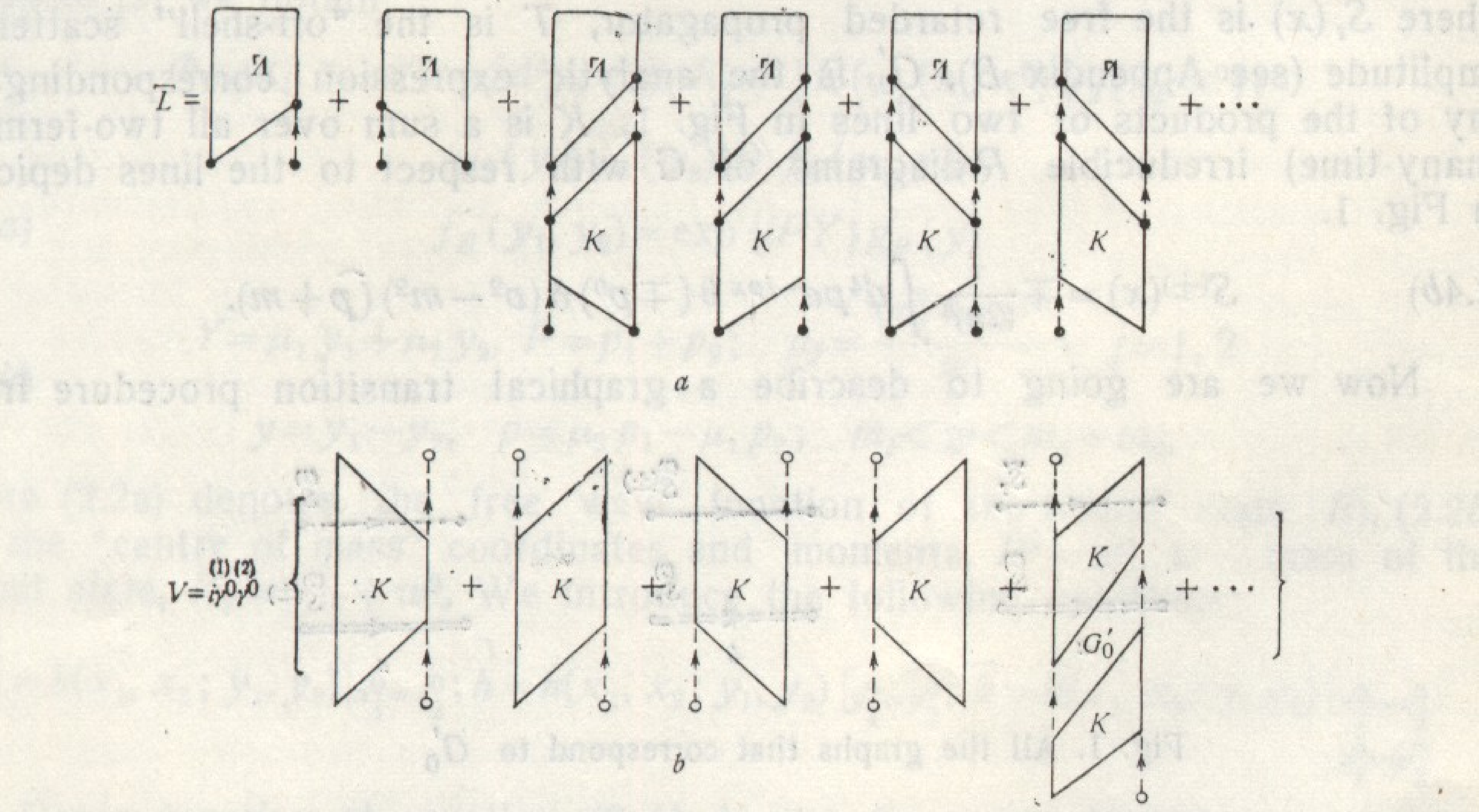


Fig. 3. Graphical expression for the kernel $\overline{L}(a)$ and for the quasipotential $V(b)$

ed under the "cutting" S_1 is $\overline{G}_0\, G^{\mathrm{Amp}} G_0 = \overline{G}.$[1] In diagrams V and VI, which are equal-time irreducible, we iterate G^{Amp} with the help of (2.4) and (2.4*b*) and then perform "cutting" of lines in the corresponding equal-time reducible dia-

[1] All blocks in Fig. 2 represent such sums.

grams, every time obtaining $\overleftarrow{G}$ after summation of all subgraphs below a fixed "cutting". The result of this successive iteration, accompanied by all possible equal-time "cuttings", is: $\overleftarrow{G}=\overleftarrow{G}_0+\overline{L}\overleftarrow{G}$ with $\overline{L}$ from (1.1) (see Fig. 3*a*). After inserting the expression $\overleftarrow{G}=\overleftarrow{G}_0+\overline{L}\overleftarrow{G}$ in (2.1) we write it down in the form

$$\Phi_{OB}=K_Y\overleftarrow{G}f_B=K_Y\overleftarrow{G}_0f_B+\overline{L}K_Y\overleftarrow{G}f_B=\overline{L}\Phi_{OB}, \tag{2.5}$$

in which the vanishing of $K_Y\overleftarrow{G}_0f_B$ has been accounted for

$$\int d^4y_1d^4y_2K_Y\overleftarrow{G}_0(\mathbf{x}_1, \mathbf{x}_2, x^0; y_1, y_2)g_B(y)e^{iPy}$$

$$=\int d^4yg_B(y)(w^2-P^2)\overleftarrow{G}_0(\mathbf{x}_1, \mathbf{x}_2, x^0; y; P)=0,$$

due to (2.2*a*) and to the fact that $\overleftarrow{G}_0$ has no pole at $P^2=w^2$.

Now we shall transform the homogeneous integral equation (2.5) in a Schrödinger-like form with a nonlocal potential. The total $\widehat{G}$ and free $\widehat{G}_0$ Green functions of (2.5) read (γ_μ are Dirac matrices):

$$\widehat{G}=\overline{G}i\,\overset{(1)}{\gamma^0}\,\overset{(2)}{\gamma^0},\quad \widehat{G}_0=\overline{G}_0i\,\overset{(1)}{\gamma^0}\,\overset{(2)}{\gamma^0}. \tag{2.6}$$

The nonlocal quasipotential V is defined by means of (see Fig. 3*b*)

$$\widehat{G}=\widehat{G}_0+\widehat{G}_0VG,\quad V=\widehat{G}_0^{-1}\overline{L} \tag{2.7}$$

or equivalently

$$V=\widehat{T}(1+\widehat{G}_0\widehat{T})^{-1},\quad \widehat{T}=\widehat{G}_0^{-1}\,\overline{G_0TG_0}i\,\overset{(1)}{\gamma^0}\,\overset{(2)}{\gamma^0}\widehat{G}_0^{-1}. \tag{2.7'}$$

Performing Fourier transform in the "centre of mass" frame $(P=P_B=(w, \mathbf{0}))$:

$$\Phi_{OB}(\mathbf{x}_1, \mathbf{x}_2, X^0)=e^{-iP_BX}\langle 0|\psi_1(x_1)\psi_2(x_2)|B\rangle=e^{-iP_BX}\Phi_w(\mathbf{x})$$

$$\Psi_w(\mathbf{p})=\frac{1}{(2\pi)^{7/2}}\int d^3x_1d^3x_2e^{-i\mathbf{p}.\mathbf{x}}\Phi_w(\mathbf{x})$$

$$V(\mathbf{p}, P_B; \mathbf{q}, Q)=\frac{1}{(2\pi)^7}\int d^3x_1d^3x_2dX^0d^3y_1d^3y_2\,dY^0\,e^{i(P_BX-QY-\mathbf{p}.\mathbf{x}+\mathbf{q}.\mathbf{y})} \tag{2.8}$$

$$\times V(\mathbf{x}, X; \mathbf{y}, Y)=\delta(P_B-Q)V_w(\mathbf{p}, \mathbf{q})$$

$$\widehat{G}_0(\mathbf{p}, P_B; \mathbf{q}, Q)=\delta(P_B-Q)\,\delta(\mathbf{p}-\mathbf{q})\,[w-\mathcal{H}_1(\mathbf{p})-\mathcal{H}_2(-\mathbf{p})+\iota o]^{-1}$$

$$\mathcal{H}_i(\mathbf{p})=\overset{(i)}{\boldsymbol{\alpha}}.\mathbf{p}+m_i\overset{(i)}{\gamma^0},\quad \boldsymbol{\alpha}=\gamma^0\boldsymbol{\gamma}.$$

Then taking into account (2.6), (2.7) we rewrite (2.5) in a quasipotential form:

$$[w-\mathcal{H}_1(\mathbf{p})-\mathcal{H}_2(-\mathbf{p})]\,\Psi_w(\mathbf{p})=\int d^3q\,V_w(\mathbf{p}, \mathbf{q})\Psi_w(\mathbf{q}). \tag{2.9}$$

3. Renormalizability of the Quasipotential

First we consider the case of two different interacting fermion fields ψ_1, ψ_2. The kernel K without any time ordering is an infinite sum of two-fermion irreducible R-diagrams, therefore it can be renormalized according to the covariant renormalization procedure for retarded functions described in [13]. Additional divergences may eventually occur in V because of the time-ordering (cf. Appendix A). However, according to the Weinberg power-counting theorem [17], the high momenta behaviour of K with respect to the external momenta is $O(\|P\|^{-2})$ in the worst case, since no diagrams of the type shown in Fig. 4, Fig. 5 appear when two different particles 1 and 2 interact. Here $\|P\|$ denotes the Euclidean norm in momentum space.

Troubles arise when treating the positronium problem. Here certain obvious changes in the wave- and Green functions (1.1), (2.3) are in order[1]

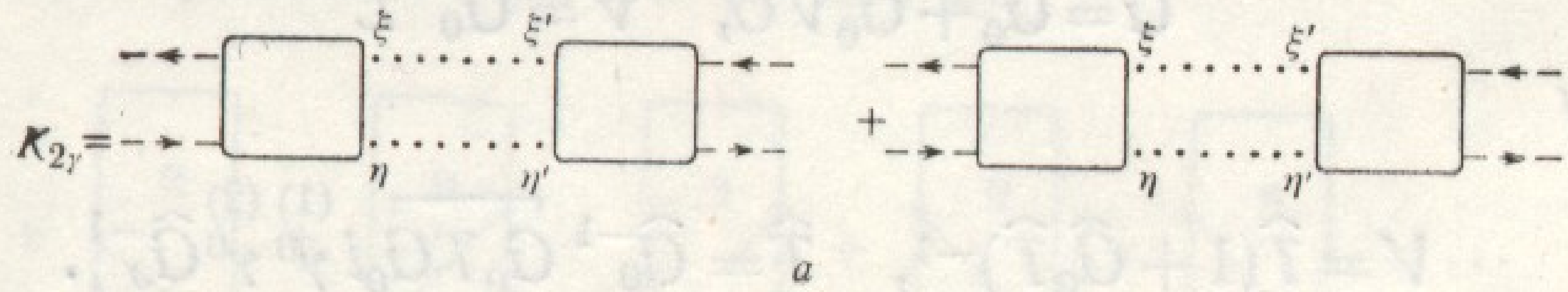
$$(3.1a)\quad \Phi_{OB}(\mathbf{x}_1, \mathbf{x}_2, t) = \langle 0 | \psi(\mathbf{x}_1, t)\psi^c(\mathbf{x}_2, t) | B\rangle,\ \psi^c(x) = \eta_c C\bar{\psi}(x), \bar{\psi}^c(x) = \bar{\psi}^c(x),$$

$$(3.1b)\quad G_{\alpha\beta\gamma\delta} = C_{\beta\beta'} C^{-1}_{\delta'\delta}\langle 0 | \psi_\alpha R(\bar{\psi}_{\beta'}\bar{\psi}_\gamma\psi_{\delta'}) + R(\psi_\alpha\bar{\psi}_\gamma\psi_{\delta'})\bar{\psi}_{\beta'}$$
$$-R(\psi_\alpha\bar{\psi}_\gamma)R(\bar{\psi}_{\beta'}\psi_{\delta'}) + R(\psi_\alpha\psi_{\delta'})R(\bar{\psi}_{\beta'}\bar{\psi}_\gamma)|0\rangle.$$

K can be represented in general as (Fig. 4a, Fig. 5a):

$$(3.1)\quad K = W + K_{2\gamma} + K_\gamma.$$

In Fig. 4[2] symmetrization with respect to the (ξ, η) and (ξ', η') vertices is assumed. In the graphs IV, V in Fig. 4b only v_1 and v_2 vertices are time-ordered. Logarithmic divergences arise in V from graphs containing graphs I, II in Fig. 4b as subgraphs because of time-ordering. The fermion loops in the graphs IV, V contain an odd number of fermion lines, hence they cancel in the sum, according to the Furry theorem. Therefore, the "dangerous" graphs I, II



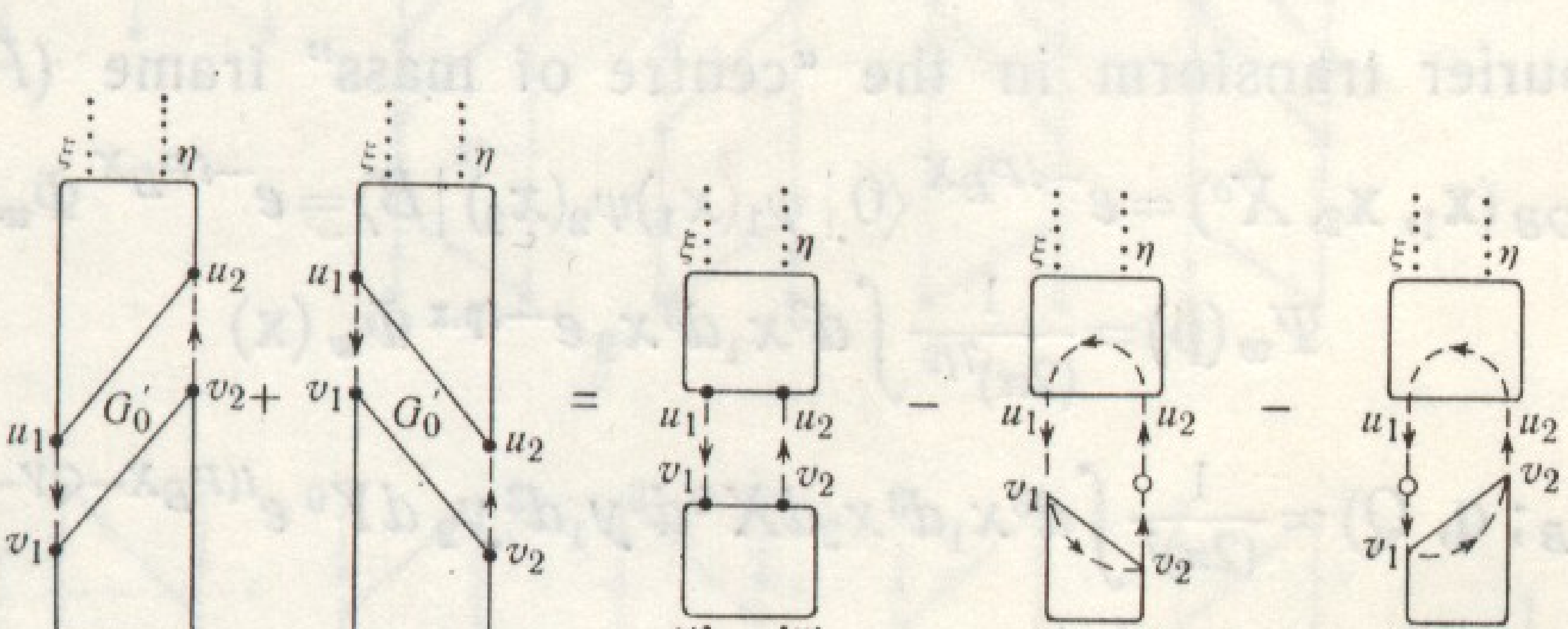


Fig. 4. Graphical expression for $K_{2\gamma}$ (a) and cancellation of divergences in it due to time ordering (b)

reduce only to graph III, which represents a sum of covariant (many-time) R-diagrams of the four-point photon retarded function, which is well defined within the framework of covariant renormalization theory.

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[1] G_0, G_0' in this case are obtained from those corresponding to different particles by replacing $S^{(2)}_r \to CS^{(1)}_r C^{-1}$, $S^{(2)(\pm)} \to CS^{(1)(\pm)}C^{-1}$, C — charge conjugation matrix, $C^{-1} = C^T$.

[2] Here and in what follows we omit the trivial factors: $C_{\beta\beta'}\ C^{-1}_{\delta'\delta}$.

The second and worst type of divergences arises from the equal-time irreducible iteration of K_γ in Fig. 3. Our proof in what follows for the renormalizability of V closely resembles that of F a u s t o v's [18] for the quasipotential in the Logunov-Tavkhelidze approach, i. e. we shall show that all divergent iterations cancel in the sum.

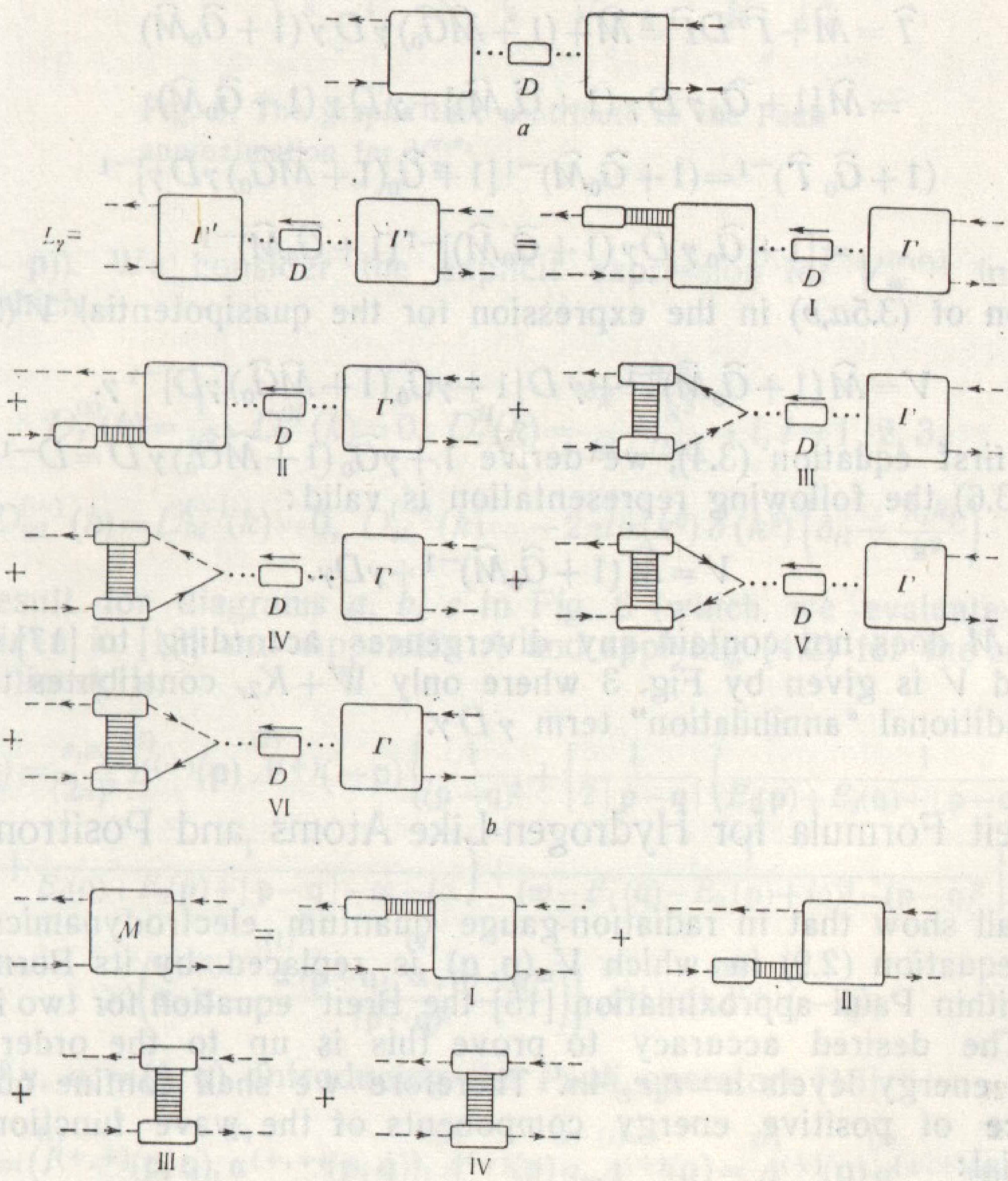

Fig. 5. Graphical expansion for the kernels $K_\gamma(a)$, $L_\gamma(b)$, $M(c)$

It is more convenient to proceed with the equivalent definition of V (2.7'). T can be decomposed into $T=M+L_\gamma$, where

(3.2) $$M=W+K_{2\gamma}+(W+K_{2\gamma})\,G_0'M,\quad L_\gamma=K_\gamma+K_\gamma\,G_0\,L_\gamma.$$

In Fig. 5 $D_{\mu\nu}=\langle 0\,|\,R(A_\mu A_\nu)\,|\,0\rangle$, $\Gamma_\mu=D_{\mu\nu}^{-1}\langle 0\,|\,R(A^\nu\psi\psi)\,|\,0\rangle$ and the shaded blocks denote sets of lines corresponding to intermediate states on the mass shell. D, Γ, Γ' satisfy the Dyson-Schwinger type of equations [19]:

(3.3) $$D=D+D\gamma G_0'\Gamma'D,\quad \Gamma=\gamma(1+G_0'M),\quad \Gamma'=(1+MG_0')\,\gamma,$$

D is the free retarded photon propagator. Applying the time-ordering procedure in (3.3) (Section 2 and Appendix A), we get

(3.4) $$D=D+D\gamma G_0\Gamma' D,\ \Gamma=\gamma(1+G_0\widehat{M}),\ \widehat{\Gamma}'=(1+\widehat{M}\widehat{G}_0)\gamma,$$
$$\widehat{\Gamma}'=\widehat{G}_0^{-1}\overset{\ulcorner}{G}_0'\Gamma',\ \widehat{\Gamma}=\Gamma \overset{\urcorner}{G_0'}\widehat{G}_0^{-1}.$$

Then applying the operation "Λ", defined in (2.7′), to (3.2) and inserting (3.4) in the result, we obtain

(3.5*a*) $$\widehat{T}=\widehat{M}+\widehat{\Gamma}' D\widehat{\Gamma}=\widehat{M}+(1+\widehat{M}\widehat{G}_0)\gamma D\gamma(1+\widehat{G}_0\widehat{M})$$
$$=\widehat{M}[1+\widehat{G}_0\gamma D\gamma(1+\widehat{G}_0\widehat{M})]+\gamma D\gamma(1+\widehat{G}_0\widehat{M}).$$

(3.5*b*) $$(1+\widehat{G}_0\widehat{T})^{-1}=(1+\widehat{G}_0\widehat{M})^{-1}[1+\widehat{G}_0(1+\widehat{M}\widehat{G}_0)\gamma D\gamma]^{-1}$$
$$=[1+\widehat{G}_0\gamma D\gamma(1+\widehat{G}_0\widehat{M})]^{-1}(1+\widehat{G}_0\widehat{M})^{-1}.$$

The insertion of (3.5*a*,*b*) in the expression for the quasipotential V (2.7′) gives

(3.6) $$V=\widehat{M}(1+\widehat{G}_0\widehat{M})^{-1}+\gamma D[1+\gamma\widehat{G}_0(1+\widehat{M}\widehat{G}_0)\gamma D]^{-1}\gamma.$$

Using the first equation (3.4), we derive $1+\gamma\widehat{G}_0(1+\widehat{M}\widehat{G}_0)\gamma D=D^{-1}D$. Therefore from (3.6) the following representation is valid:

$$V=\widehat{M}(1+\widehat{G}_0\widehat{M})^{-1}+\gamma D\gamma.$$

Apparently $\widehat{M}$ does not contain any divergences according to [17]. Thus the renormalized V is given by Fig. 3 where only $W+K_{2\gamma}$ contributes to K (3.1*c*) with the additional "annihilation" term $\gamma D\gamma$.

4. Breit Formula for Hydrogen-Like Atoms and Positronium

We shall show that in radiation-gauge quantum electrodynamics the quasipotential equation (2.9) in which $V_w(\mathbf{p},\mathbf{q})$ is replaced by its Born term, reproduces within Pauli approximation [15] the Breit equation for two interacting particles. The desired accuracy to prove this is up to the order α^4 of the bound-state energy levels $\alpha=e_1e_2/4\pi$. Therefore we shall confine ourselves to the subspace of positive energy components of the wave function and the quasipotential:

$$\Psi_w^{(+,+)}(\mathbf{p})=\overset{(1)}{\Lambda}{}^{(+)}(\mathbf{p})\overset{(2)}{\Lambda}{}^{(+)}(-\mathbf{p})\,\Psi_w(\mathbf{p}),\quad \overset{(i)}{\Lambda}{}^{(+)}(\mathbf{p})=\frac{E_i(\mathbf{p})+\mathcal{H}_i(\mathbf{p})}{2E_i(\mathbf{p})}$$

$$V_w^{(++,++)}(\mathbf{p},\mathbf{q})=\overset{(1)}{\Lambda}{}^{(+)}(\mathbf{p})\overset{(2)}{\Lambda}{}^{(+)}(-\mathbf{p})V_w(\mathbf{p},\mathbf{q})\overset{(1)}{\Lambda}{}^{(+)}(\mathbf{q})\overset{(2)}{\Lambda}{}^{(+)}(-\mathbf{q}),\ E_i(\mathbf{p})=\sqrt{m_i^2+\mathbf{p}^2},$$

where $\mathcal{H}_i(\mathbf{p})$ is from (2.8), $\overset{(i)}{\Lambda}(\mathbf{p})$, $i=1, 2$ are the positive energy projection operators. Then (2.9) acquires the form

(4.1) $$[w-E_1(\mathbf{p})-E_2(\mathbf{p})]\,\Psi_w^{(+,+)}(\mathbf{p})=\int d^3q\, V_w^{(++,++)}(\mathbf{p},\mathbf{q})\,\Psi_w^{(+,+)}(\mathbf{q}).$$

After applying the equal-time transition procedure described in Section 2, we obtain the following time-ordered R-diagrams for the terms proportional to e_1e_2 in $V_w(\mathbf{p},\mathbf{q})$ perturbation theory expansion (Fig. 3*b*) ($V_w^{(e_1e_2)}$):

Here we have ignored the disconnected diagram corresponding to the term

$$\langle 0|R(\psi_1\overline{\psi}_1)|0\rangle\langle 0|R(\psi_2\overline{\psi}_2)|0\rangle$$

in (2.3), since their contribution to the energy level corrections is of an order higher than α^4 [20]. The diagram in Fig. 6*d* has no contribution either to the desired accuracy because the $-iS^{(2)(+)}$-line gives the negligible term

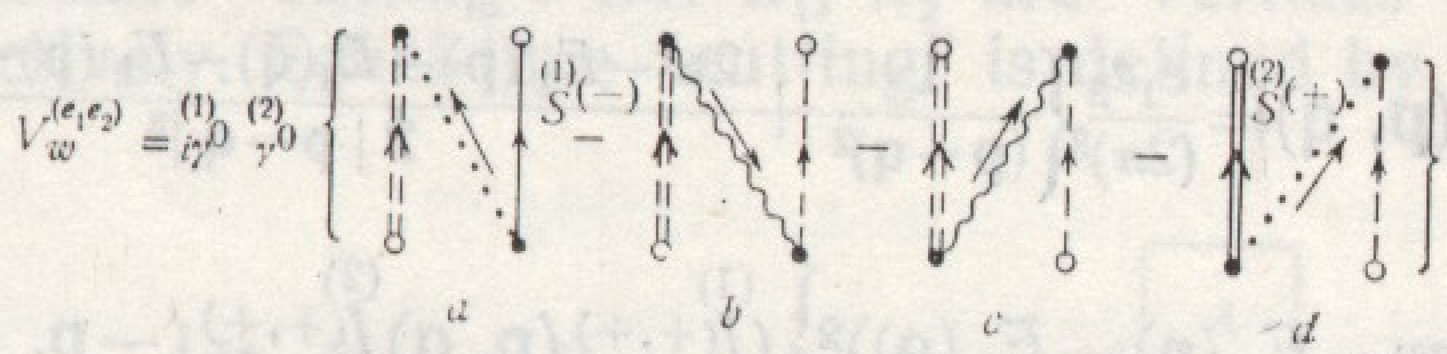

Fig. 6. The graphs that contribute to the Pauli approximation for $V_w^{e_1e_2}$

$(\sim \Lambda^{(2)(-)}(-\mathbf{p}))$. We consider the explicit expression for $V_w^{(e_1e_2)}$ in radiation gauge, in which

$$D_r^{00}(k)=\frac{1}{\mathbf{k}^2},\ D_r^{0i}(k)=0,\ D_r^{il}(k)=\frac{\delta_{il}-\frac{k_ik_l}{\mathbf{k}^2}}{k^2+ik^00},\ i,l=1,2,3, \tag{4.2}$$

$$D_{00}^{(-)}(k)=D_{0i}^{(-)}(k)=0,\ D_{il}^{(-)}(k)=-2\pi i\theta(k^0)\,\delta(k^2)\left(\delta_{il}-\frac{k_ik_l}{\mathbf{k}^2}\right).$$

The final result for diagrams *a*, *b*, *c* in Fig. 6 (which we evaluate following the rules given in [13] and Appendix *A* and applying (4.2) for the corresponding photon lines), is

$$V_w^{(e_1e_2)}(\mathbf{p},\mathbf{q})=\frac{e_1e_2}{(2\pi)^3}\overset{(1)}{\Lambda}{}^{(+)}(\mathbf{p})\overset{(2)}{\Lambda}{}^{(+)}(-\mathbf{p})\Big\{\frac{1}{(\mathbf{p}-\mathbf{q})^2}+\Big[\frac{1}{2\,|\,\mathbf{p}-\mathbf{q}\,|}\Big(\frac{1}{E_1(\mathbf{p})+E_2(\mathbf{q})-|\,\mathbf{p}-\mathbf{q}\,|-w-io}$$

$$+\frac{1}{E_1(\mathbf{q})+E_2(\mathbf{p})+|\,\mathbf{p}-\mathbf{q}\,|-w-io}\Big)+\frac{1}{(w-E_1(\mathbf{q})-E_2(\mathbf{p})+io)^2-(\mathbf{p}-\mathbf{q})^2}\Big] \tag{4.3}$$

$$\times\Big(\overset{(1)}{\boldsymbol{\alpha}}\cdot\overset{(2)}{\boldsymbol{\alpha}}-\frac{\overset{(1)}{\boldsymbol{\alpha}}(\mathbf{p}-\mathbf{q})\,.\,\overset{(2)}{\boldsymbol{\alpha}}\,.\,(\mathbf{p}-\mathbf{q})}{(\mathbf{p}-\mathbf{q})^2}\Big)\Big\}\overset{(1)}{\Lambda}{}^{(+)}(\mathbf{q})\overset{(2)}{\Lambda}{}^{(+)}(-\mathbf{q}),$$

where $\boldsymbol{\alpha}=\gamma^0\boldsymbol{\gamma}$, $\alpha_\mu=(1,\boldsymbol{\alpha})$. Introducing the Pauli operators [15]

$$\overset{(i)}{\alpha}{}_\mu^{(+,+)}(\mathbf{p},\mathbf{q})=(\overset{(i)}{I}{}^{(+,+)}(\mathbf{p},\mathbf{q}),\overset{(i)}{\boldsymbol{\alpha}}{}^{(+,+)}(\mathbf{p},\mathbf{q})),\ \overset{(i)}{\Lambda}{}^{(+)}(\mathbf{p})\,\overset{(i)}{\alpha}_\mu\overset{(i)}{\Lambda}{}^{(+)}(\mathbf{q})=\overset{(i)}{\Lambda}{}^{(+)}(\mathbf{p})\,\overset{(i)}{\alpha}{}_\mu^{(+,+)}(\mathbf{p},\mathbf{q}),\ i=1,2,$$

the quasipotential (4.3) in the semirelativistic Pauli approximation [15] becomes local and has the following form:

$$V_w^{(e_1e_2)}(\mathbf{p},\mathbf{q})=\frac{e_1e_2}{(2\pi)^3}\frac{1}{(\mathbf{p}-\mathbf{q})^2}\Big\{\overset{(1)}{I}{}^{(+,+)}(\mathbf{p},-\mathbf{q})\overset{(2)}{I}{}^{(+,+)}(-\mathbf{p},-\mathbf{q})-\overset{(1)}{\boldsymbol{\alpha}}{}^{(+,+)}(\mathbf{p},\mathbf{q})\,.\,\overset{(2)}{\boldsymbol{\alpha}}{}^{(+,+)}(-\mathbf{p},-\mathbf{q}) \tag{4.4a}$$

$$+\frac{1}{(\mathbf{p}-\mathbf{q})^2}[\overset{(1)}{\boldsymbol{\alpha}}{}^{(+,+)}(\mathbf{p},\mathbf{q})\,.\,(\mathbf{p}-\mathbf{q})]\,[\overset{(2)}{\boldsymbol{\alpha}}{}^{(+,+)}(-\mathbf{p},-\mathbf{q})\,.\,(\mathbf{p}-\mathbf{q})]\Big\},$$

which exactly coincides with the Breit potential [15].

After the same calculations, now performed in Feynman gauge, the quasipotential reads:

$$V_{w,\mathrm{Feyn}}^{(e_1e_2)}(\mathbf{p},\mathbf{q})=\frac{e_1e_2}{(2\pi)^3}\Big\{\frac{1}{(\mathbf{p}-\mathbf{q})^2-(w-E_1(\mathbf{p})-E_2(\mathbf{q})+i0)^2}-\frac{1}{2\,|\,\mathbf{p}-\mathbf{q}\,|} \tag{4.4b}$$

$$\times\Big[\frac{1}{E_1(\mathbf{q})+E_2(\mathbf{p})+|\,\mathbf{p}-\mathbf{q}\,|-w-i0}+\frac{1}{E_1(\mathbf{p})+E_2(\mathbf{q})-|\,\mathbf{p}-\mathbf{q}\,|-w-i0}\Big]\Big\}$$

$$(\overset{(1)}{I}{}^{(+,+)}(\mathbf{p}, \mathbf{q})\, \overset{(2)}{I}{}^{(+,+)}(-\mathbf{p}, -\mathbf{q}) - \overset{(1)}{\boldsymbol{\alpha}}{}^{(+,+)} \cdot \overset{(2)}{\boldsymbol{\alpha}}{}^{(+,+)}).$$

or, in Pauli approximation

$$\text{(4.4c)} \quad V^{(e_1 e_2)}_{w,\mathrm{Feyn}}(\mathbf{p}, \mathbf{q}) = \frac{e_1 e_2}{(2\pi)^3} \Big\{ \frac{1}{(\mathbf{p}-\mathbf{q})^2} + \frac{2w - E_1(\mathbf{p}) - E_1(\mathbf{q}) - E_2(\mathbf{p}) - E_2(\mathbf{q})}{2\,|\mathbf{p}-\mathbf{q}|^3}$$

$$+ \frac{1}{|\mathbf{p}-\mathbf{q}|^4}(w - E_1(\mathbf{p}) - E_2(\mathbf{q}))^2 \Big\} (\overset{(1)}{I}{}^{(+,+)}(\mathbf{p}, \mathbf{q})\, \overset{(2)}{I}{}^{(+,+)}(-\mathbf{p}, -\mathbf{q})$$

$$- \overset{(1)}{\boldsymbol{\alpha}}{}^{(+,+)}(\mathbf{p}, \mathbf{q}) \cdot \overset{(2)}{\boldsymbol{\alpha}}{}^{(+,+)}(-\mathbf{p}, -\mathbf{q})),$$

which obviously does not reproduce the Breit potential. However, the second term in the parenthesis of (4.4*c*) that is responsible for the spurious α^3 corrections to the energy levels cancels with the α^3 contribution of all equal-time R-diagrams for $V_{w,\mathrm{Feyn}}$ proportional to $e_1^2 e_2^2$ [21].

In the case of positronium ($e_1 = -e_2 = e$, $m_1 = m_2 = m$) we have an additional term $\gamma D \gamma$ in V (3.7) within the desired approximation which gives the well-known "annihilation" corrections to the positronium energy levels [19].

5. Conclusion

The main objective of the present work was to give a consistent renormalizable quasipotential formalism based on the fundamental structure of quantum field theory: reduction formulae and covariant renormalization theory. Unlike the Logunov-Tavkhelidze and equal-time Bethe-Salpeter singular free Green function for the quasipotential equation in the case of particles of spin 1/2, $\widehat{G}_0$ from (2.8) is nonsingular. The equal-time transition procedure (Section 2) looks here simpler than the corresponding one for the equal-time Bethe-Salpeter equation [8], due to the intrinsic time-ordering of vertices by S_r and $\bar{S}_r$ in R-diagrams.

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Appendix A. Graphical Rules for Time-ordered R-Diagrams

1. Time-ordering. A subset of vertices belonging to a given R-diagram (described in [13]) on which time conditions (time-ordering) are imposed, is graphically represented by placing the vertices in upward order (or in horizontal order from right to left), according to the time increase.

2. "Cutting" of lines. To each S_r, $S^{(\pm)}$, ψ^{in}, etc. line, two lines correspond that are divided by an additional "cutting" vertex $i\gamma^0$ with respect to the equations [22]

$$\psi^{in}(x)^{(\pm)} = -i \int d^3z S^{(\pm)}(x-z)\, \gamma^0\, \psi^{in(\pm)}(z),$$

$$\text{(A.1)} \quad S^{(\mp)}(x-y) = -i \int d^3z S^{(\mp)}(x-z)\, \gamma^0 S^{(\mp)}(z-y) = -i \int_{z^0 < x^0 (z^0 > x^0)} d^3z \overset{(-)}{S_r}(x-z)\, \gamma^0\, S^{(\mp)}(z-y),$$

$$S_{\rm r}(x-y) = -i \int\limits_{x^0>y^0>z^0} d^3z\, S_{\rm r}(x-z)\, \gamma^0\, S_{\rm r}(z-y).$$

They are graphically represented in Fig. 7*a*.

3. Equal-time "cutting". Let x_1, x_2 are vertices of two different (fermion) paths, respectively. Equal-time "cutting" is defined by the time condition $x_1^0 = x_2^0$.

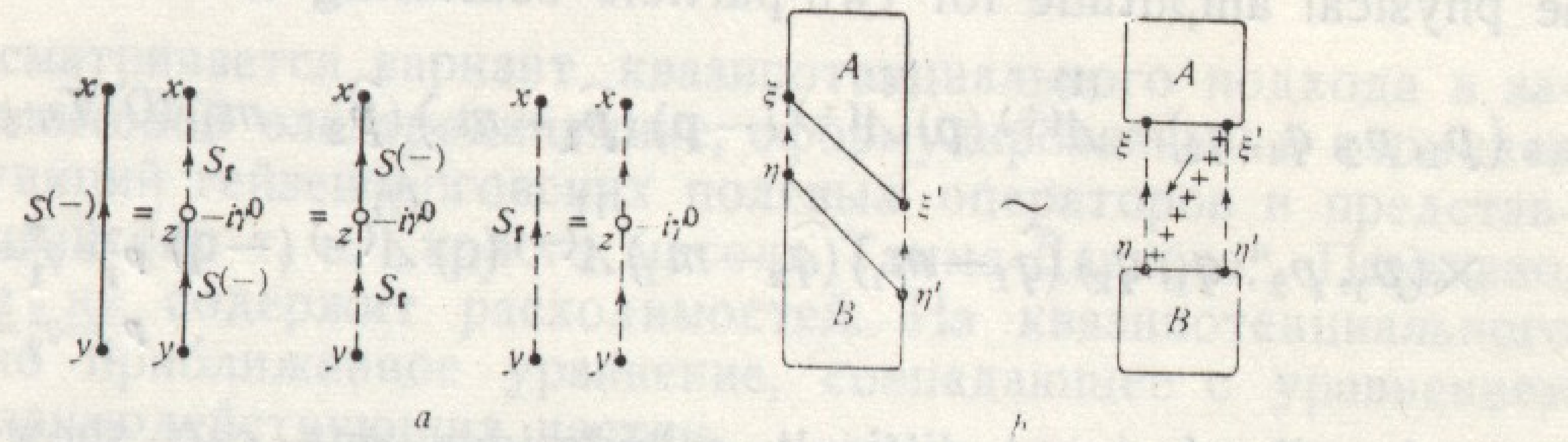

Fig. 7. Graphical representation of eqns. (*A*. I) (*a*) and the time-ordering procedure (*b*)

4. Two-fermion equal-time irreducible R-diagrams are those for which no equal-time "cutting" exists. An example is shown in Fig. 7*b*, I.

5. Time conditions in momentum space.

a) equal-time "cutting" ($x_1^0 = x_2^0$) corresponds to integration over the relative energy p^0 (2.2*b*).

b) time-ordering of the vertices ξ, η, ξ', η' (Fig. 7*b*, I) corresponds to an additional line $[2\pi i(\omega - io)]^{-1}$ of a ficticious particle with four momentum $\widetilde{\omega} = (\omega, o)$ in Fig. 7*b*, II without time conditions.

Appendix *B*. Quasipotential Off-Shell Scattering Amplitude

We consider the wave function Φ_{OS} of two particles in the case of scattering and use the usual reduction formulae [16] for the scattering state $|S\rangle$:

$$\Phi_{OS}(\mathbf{x}_1, \mathbf{x}_2, X^0) = \langle 0 | \psi_1(\mathbf{x}_1, X^0)\, \psi_2(\mathbf{x}_2, X^0) | S \rangle = \int d^4y_1\, d^4y_2 \langle 0 | R(\psi_1(\mathbf{x}_1, X^0)$$

$$\times \psi_2(\mathbf{x}_2, X^0);\ \overline{\psi}_1(y_1)\, \overline{\psi}_2(y_2) | 0 \rangle \overline{D}^{(1)}_{y_1} \overline{D}^{(2)}_{y_2} \overset{(1)}{u}_{\sigma_1}(\mathbf{q}_1)\, \overset{(2)}{u}_{\sigma_2}(\mathbf{q}_2)\, e^{-iq_1y_1 - iq_2y_2}; \tag{B.1}$$

$$\overline{D}^{(j)}_x = i \overset{(j)\leftarrow}{\gamma^\mu} \frac{\partial}{\partial x^\mu} + m_j,\ j = 1, 2;\ |S\rangle = b^{(1)*in}_{\sigma_1}(\mathbf{q}_1)\, b^{(2)*in}_{\sigma_2}(\mathbf{q}_2) | 0 \rangle.$$

Here $b^{(j)*in}$, $\overset{(j)}{u}_{\sigma_j}$, q_j denote creation "in" operators, free wave functions and momenta of the particles, respectively. After insertion of $\ulcorner G = \ulcorner G_0 + \ulcorner G'_0\, T G_0$ in (*B*.1) we get

$$\Phi_{OS} = \Phi^{(0)}_{OS} + \ulcorner G'_0\, T \Psi^{(0)}_{OS};\quad \Psi^{(0)}_{OS} = \langle 0 | \psi_1^{in}(x_1)\, \psi_2^{in}(x_2) | S \rangle,\ \Phi^{(0)}_{OS} = \ulcorner \Psi^{(0)}_{OS}. \tag{B.2}$$

Making use of (*A*.1) we obtain from (*B*.2) the quasipotential Lippmann-Schwinger equation

$$\Phi_{OS} = \Phi^{(0)}_{OS} + \widehat{G}_0\, \widehat{\mathfrak{R}}\, \Phi_{OS},$$

where $\widehat{G}_0$ is introduced in (2.6) and the quasipotential scattering (off-shell) amplitude $\widehat{\mathfrak{R}}$ is defined as

$$\widehat{\mathfrak{R}}(X^0, \mathbf{x}_1, \mathbf{x}_2;\ Y^0, \mathbf{y}_1, \mathbf{y}_2) = i \overset{(1)}{\gamma^0} \overset{(2)}{\gamma^0} \sum_{j,k=1,\ k\neq j}^{2} \int d^4z_1\, d^4z_2\, d^4u\, \overset{(j)}{S_r}(x_j - u)$$

$$\times T(x_k, u;\, z_1, z_2)\, \overset{(1)}{S^{(-)}}(z_1 - y_1)\, \overset{(2)}{S^{(-)}}(z_2 - y_2)\, \overset{(1)}{\gamma^0} \overset{(2)}{\gamma^0} \Big|_{\substack{x_1^0 = x_2^0 = X^0 \\ y_1^0 = y_2^0 = Y^0}}.$$

The physical amplitude for two-particle scattering is

$$T_{\mathrm{Phys}}(p_1, p_2, q_1, q_2) = \overset{(1)}{\Lambda^{(+)}}(\mathbf{p})\, \overset{(2)}{\Lambda^{(+)}}(-\mathbf{p})\, (\widehat{p}_1 - m_1)(\widehat{p}_2 - m_2) \langle 0\, |\, T\psi_1\psi_2\bar{\psi}_1\bar{\psi}_2\, |\, 0\rangle$$

$$\times (p_1, p_2;\, q_1, q_2)(\widehat{q}_1 - m_1)(\widehat{q}_2 - m_2)\, \overset{(1)}{\Lambda^{(+)}}(\mathbf{q})\, \overset{(2)}{\Lambda^{(+)}}(-\mathbf{q}) \Big|_{\substack{p_1^2 = q_1^2 = m_1^2 \\ p_2^2 = q_2^2 = m_2^2}}.$$

After some lengthy but not difficult calculations one can show [21] that the positive energy component of the Fourier transform of $\mathfrak{R} - \widehat{\mathfrak{R}}^{(++,++)}$ on the mass and energy shell coincides with T_{Phys}. An analogous result within the framework of Logunov-Tavkhelidze's approach was obtained in [12].

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Нелокальное квазипотенциальное уравнение в терминах запаздывающих функций

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(Резюме)

В работе рассматривается вариант квазипотенциального подхода в задаче двух тел в квантовой электродинамике, сформулированный в терминах запаздывающих функций гейзенберговских полевых операторов и представляющий определенный тип т. н. „нового метода Тамма-Данкова". Показано, что квазипотенциал не содержит расходимостей. Из квазипотенциального уравнения выведено приближенное уравнение, совпадающее с уравнением Брейта для двух взаимодействующих частиц.

Infinite Set of Conservation Laws in the Quantum Sine-Gordon and the Massive Thirring Models

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The infinite set of conserved currents in the Sine-Gordon and the massive Thirring field-theory models is considered. In the quantum theory certain anomalous terms in the Ward identities arise as a result of renormalization. It is shown that the former lead actually only to a slight modification of the classical expressions for the conserved currents. Thus the most remarkable features of the models: particle number conservation, trivial scattering and factorization of the S-matrix are proved in perturbation theory.

1. Introduction

In the last few years some exact solvable nonlinear evolution equations [1] attracted extreme interest in particle physics since they exhibit nontrivial "extended particle" solutions (solitons), which may provide a fundamental basis for constructing unified theories where hadrons arise as coherent excitations of lepton fields [2]. In this sense two dimensional models are known to be of great significance in elucidating the phenomenon of solitons. The most interesting examples are the Sine-Gordon (SG) and the massive Thirring (MTh) models with Lagrangians, respectively,

$$\mathscr{L}^{\mathrm{SG}}=\frac{1}{2}(\partial_\mu\varphi)^2+\frac{m^2}{\beta^2}(\cos\beta\varphi-1),\quad \mathscr{L}^{\mathrm{MTh}}=\frac{i}{2}\bar{\psi}\gamma^\mu\overleftrightarrow{\partial_\mu}\psi-m\bar{\psi}\psi-\frac{\lambda}{4}(\bar{\psi}\gamma^\mu\psi)^2.$$

At the classical level they were exactly solved by the inverse scattering method [3, 4] and were shown to be completely integrable Hamiltonian sys-